ROTATIONAL MOTION

Preface

Either it be shrilling spin balling or a rejuvenating beledance, there is concept of rotational motion. Though the motion of rigid body seems to be tedious and confusing but in this textual material even an trivial concept is substantialised in this way to make it lucid and easy to grasp. Before starting this chapter you should be aware of basic concepts of motion and after accomplishment of this chapter you must be able to

- Visualize motion of Rigid Bo dy
- Understand physical significance of MI and its calculation
- Know about concept of torque which is significantly significant feature of rotational motion
- Develope the concept of conservation of energy in rotation motion
- Familiar with problem solving tactics

Go through the text review questions and all exercise to make yourself, perfect in attacking on AIEEE problems

This book consists of theoritical & practical explanations of all the concepts involved in the chapter. Each article followed by a ladder of illustration. At the end of the theory part, there are miscellaneous solved examples which involve the application of multiple concepts of this chapter.

Students are advised to go through all these solved examples in order to develope better understanding of the chapter and to have better grasping level in the class.

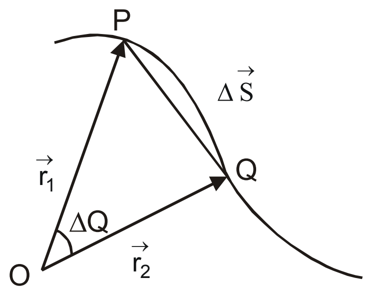
Total number of Questions in **Rotational Motion** are :

In Chapter Examples 29

Solved Examples 36

Total no. of questions 65

1. ANGULAR DISPLACEMENT



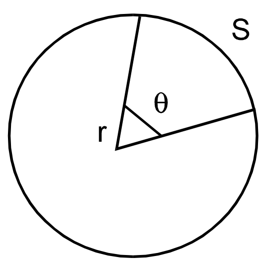
a. When a particle moves in a curved path, the angle subtended by its position vector about a fixed point is known as angular displacement.

b. **Unit** : radian

c. **Dimension** : $M^0L^0T^0$

d. It is a vector quantity whose direction is given by right hand screw rule.

e. If θ be the angular displacement of a particle moving in a circular path of



radius r and S be the length of arc, $S = r\theta$

Note :- $360^\circ = 2\pi$ radian

Example based on Angular displacement

Ex.1 If a particle complete one and half revolution along the circumference of a circle then its angular displcement is-

(A) 0 (B) π
(C) 2π (D) 3π

Sol.(D) = 3π

2. ANGULAR VELOCITY

a. The angular displacement per unit time is defined as angular velocity. If, a particle moves from P to Q in time Δt, $\omega = \frac{\Delta\theta}{\Delta t}$, where $\Delta\theta$ is the angular displacemnet during the time interval Δt.

If $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$ = Instantaneous angular velocity .

$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$ = average angular velocity

b. Unit : Radian /sec

c. Dimension : $[M^0L^0T^{-1}]$ which is same as that of frequency.

d. It is a vector quantity, whose direction is normal to the rotational plane & its direction is given by right hand screw rule.

e. If ω be the angular velocity, v be the linear velocity & r be the radius of path, we have the following relation. $\vec{v} = \vec{\omega} \times \vec{r}$

or $v = r\omega$.

f. If f be the frequency, $\omega = 2f\pi$

If T be the time period , $\omega = \frac{2\pi}{T}$

g. The angular veloctiy of a rotating rigid body can be either positive or negative, depending on whether body is rotating in the direction of increasing θ (anticlokwise) or of decreasing θ (clockwise) .

h. The magnitude of an angular velocity is called the angular speed which in also represented by ω,

1.1 Equation of Linear Motion and Rotational Motion

Linear Motion	Rotational Motion
(a) If acceleration is 0, v = constant and s = vt.	(a) If acceleration is 0, ω = constant and θ = ωt.
(b) If acceleration a = constant,	(b) If acceleration a = constant then
(i) $s = \frac{(u+v)}{2}\,t$	(i) $\theta = \frac{(\omega_1+\omega_2)}{2}\,t$
(ii) $a = \frac{v-u}{t}$	(ii) $\alpha = \frac{\omega_2-\omega_1}{t}$
(iii) v = u+ at	(iii) $\omega_2 = \omega_1 + \alpha t$
(iv) $s = ut + (1/2)\,at^2$	(iv) $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
(v) $v^2 = u^2 + 2as$	(v) $\theta = \omega_1^2 + 2\alpha\theta$
(vi) $S_{nth} = u + \frac{1}{2}a\,(2n-1)$	(vi) $\theta_{nth} = \omega_1 + (2n-1)\frac{\alpha}{2}$
(c) If acceleration is not constant, the above equation will not be applicable. In this case	(c) If acceleration is not constant, the above equation will not be applicable. In this case
i) $v = \frac{dx}{dt}$	i) $\omega = \frac{d\theta}{dt}$
ii) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	ii) $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
iii) v dv = a ds	iii) $\omega\, d\omega = \alpha\, d\theta$

Examples based on **ANGULAR VELOCITY**

Ex.2 A wheel of mass 6 kg is rotating at 300 rpm. Its angular velocity will be-
(A) 31.4 rad/sec (B) 3.14 rad/sec
(C) 0.314 rad/sec (D) 0.0314 rad/sec

Sol.(A) Here, $\omega = \frac{2\pi n}{t} = \frac{2\times3.14\times300}{60}$ = 31.4 rad/sec

Ex.3 If angular displacement of a particle moving on a curved path be given as, $\theta = 1.5\,t + 2t^2$, where t is in sec, the angular velocity at t = 2 sec, will be- (in rad/sec)
(A) 1.5 (B) 2.5 (C) 9.5 (D) 8.5

Sol.(C) $\theta = 1.5\,t + 2t^2$, $\therefore \frac{d\theta}{dt} = 1.5 + 4t$,

Now, $\left(\frac{d\theta}{dt}\right)_{at\,t=2\,sec} = 1.5 + 4 \times 2 = 1.5 + 8$

= 9.5 rad/sec

3. ANGULAR ACCELERATION

(a) The rate of change of angular velocity is defined as angular acceleration.

(b) Suppose at time t_1, a particle has angular velocity $\vec{\omega}_1$ & $\vec{\omega}_2$ at time t_2 then angular acceleration, $\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$

$\Rightarrow \vec{\alpha} = \frac{d^2\theta}{dt^2}$

(c) It is a vector quantity, whose direction is along the change in direction of angular velocity.

(d) **Unit :** Radian/sec^2

(e) **Dimension :** $M^0L^0T^{-2}$

(f) Relation between angular acceleration $\vec{\alpha}$ and linear acceleration $\vec{a}$, is

$\vec{a} = \vec{\alpha} \times \vec{r}$ or $a = \alpha r$

(g) Radial acceleration, $\vec{a}_r = \vec{\omega} \times \vec{v}$. Its direction is along the radius.

(h) Angular velocity, angular acceleration etc are related to rotational axis and are known as axial vector.

Examples based on Angular acceleration

Ex.4 A fly wheel starting from rest acquires in 10 sec an angular velocity of 240 revolutions per minute. The angular acceleration will be-

(A) 25.1 rad/sec^2 (B) 2.51 rad/sec^2
(C) 0.251 rad/sec^2 (D) 251 rad/sec^2

Sol.(B) ω_1 = 0 rad/sec, $\omega_2 = 2\pi$ (240/60)
= 25.1 rad/sec

$$\therefore \quad \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{25.1 - 0}{10} = 2.51 \text{ rad/sec}^2$$

Ex.5 The angular velocity of a wheel increases from 1200 to 4500 rev/min in 10 sec. The number of revolutions made during this time is-

(A) 475 (B) 425 (C) 450 (D) 950

Sol.(A) We know that $\omega = \omega_0 + \alpha t$

Here, ω_0 = 1200/60 = 20 rev/sec = 40π rad/sec

and ω = 4500/60 = 75 rev/sec = 150 π rad/sec

$\therefore$ $150\pi = 40\pi + \alpha(10)$

$$\Rightarrow \quad \alpha = \frac{(150-40)\pi}{10} = 11\pi \text{ rad/sec}^2$$

Let θ be the angle traced by wheel during this time, then

$\theta = \omega_0 t + (1/2)\alpha t^2$

$= 40\pi \times 10 + (1/2) \times 11\pi (10)^2 = 950\pi$

Now number of revolutions n is given by,

$$n = \frac{\theta}{2\pi} = \frac{950\pi}{2\pi} = 475$$

Ex.6 A car is moving at a speed of 72 km/hr. The diameter of its wheel is 0.50 m. If the wheels are stopped in 20 rotations by applying brakes, the angular retardation produced by the brakes will be-

(A) 25.5 rad/s^2 (B) 2.55 rad/s^2
(C) 0.255 rad/s^2 (D) 255 rad/s^2

Sol.(A) The speed of the car is,

$$v = 72 \text{ km/h} = \frac{72 \times 1000}{60 \times 60} = 20 \text{ m/s}$$

and the radius of each wheel is,

r = 0.25 m

Therefore, the angular speed of the wheel is,

$$\omega = \frac{v}{r} = \frac{20}{0.25} = 80 \text{ rad/s}$$

The angular displacement in 20 rotations is,

$\theta = 2\pi \times 20 = 40\pi$ radian

From $\omega^2 = \omega_0^2 + 2\alpha\theta$, we have

$0 = (80)^2 + 2\alpha(40\pi)$

$\Rightarrow \quad \alpha = -25.5$ rad/s^2

The negative sign means that there is retardation.

Ex.7 A wheel starting from rest, is rotating with a constant angular acceleration of 3.0 rad/sec^2. An observer notes that it traces an angle of 120 radian in 4.0 sec interval. For how long the wheel had been rotated when the observer started his observation ?

(A) 4 sec (B) 2 sec
(C) 8 sec (D) 6 sec

Sol.(C) Let ω_0 be the angular velocity of wheel, when the observer started his observation.

Now, $\theta = \omega_0 t + (1/2)\alpha t^2$

Substituting the values, we get

$120 = \omega_0 \times 4.0 + (1/2) \times 3.0 \times (4.0)^2$

$\Rightarrow \omega_0 = 24$ rad/sec

Further, $\omega_0 = \omega_0' + \alpha t$ ......(1)

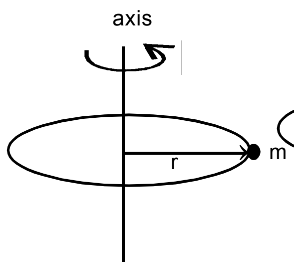
Where ω_0' is the initial angular velocity at the starting of motion

Here $\omega_0 = 24$ rad/sec

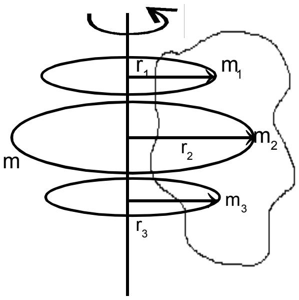
$\omega'_0 = 0$ and $\alpha = 3.0$ rad/sec^2

$\therefore$ t = 8.0 sec [from (1)]

4. MOMENT OF INERTIA (ROTATIONAL INERTIA)



axis of rotation



axis of rotation

(a) The virtue, by which a body revolving about an axis opposes the change in rotational motion, is known as moment of inertia.

(b) The moment of inertia of a particle with respect to an axis of rotation is equal to the product of mass of the particle and square of distance from the axis, hence $I = mr^2$

(c) The moment of inertia of a system about an axis of rotation is equal to the sum of moment of inertia of all the particles of the system about the axis of rotation.

$$I = m_1r_1^2 + m_2r_2^2 + \text{------} = \sum m_ir_i^2 = \int r^2 dm$$

(d) It is a scalar quantity

(e) **Unit :** In M.K.S = kg - m^2,

In C.G.S = gm-cm^2

(f) **Dimension :** $[M^1L^2T^0]$

(g) Moment of inertia depends on the following factors.

i) Mass of body

ii) Mass distribution of body or shape, size, density of body.

iii) On the position of axis of rotation.

Note : The more is the distribution of mass with respect to axis of rotation the more will be moment of inertia.

(h) Moment of inertia does not depend on the following factors.

i) Angular velocity (ω)

ii) Angular Acceleration (α)

iii) Torque (τ)

vi) Angular Momentum (J)

4.1 Radius of Gyration - (K)

(a) The distance, from the axis of rotation where, the entire mass of the body is supposed to be concentrated and the value of moment of inertia is same as that due to actual distribution of masses of body, is called radius of gyration.

(b) The radius of gyration of a body about different axes is different

(c) If K be the radius of gyration, $I = mK^2$

$$\Rightarrow K = \sqrt{\frac{I}{m}}$$

$$\Rightarrow K = \sqrt{\frac{m_1r_1^2 + m_2r_2^2 + --- m_nr_n^2}{m_1 + m_2 + --- + m_n}}$$

(d) For a symmetrical body, the radius of gyration is equal to the root mean square of distances of all the particles from the axis of rotation.

or if, $m_1 = m_2 = \text{-----------} = m_n$

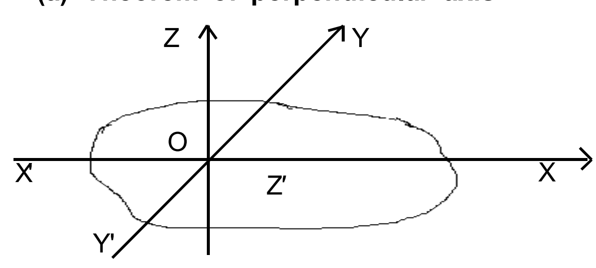
$$K = \sqrt{\frac{r_1^2 + r_2^2 + --- + r_n^2}{n}} = r_{rms}$$

(e) The value of radius of gyration depends upon the axis of rotation and mass distribution with respect to it

(f) Radius of gyration does not depend upon mass of body.

4.2 Theorems of moment of inertia

(a) Theorem of perpendicular axis -

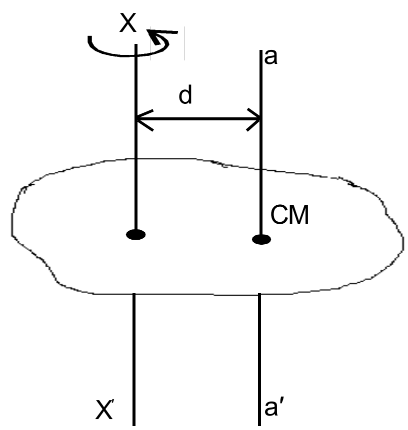


According to this theorem the moment of inertia of a lamina about an axis passing through its axis and perpendicular to its plane is equal to the sum of moment of inertia about the two mutually perpendicular axis in the plane of lamina. The normal axis OZ must pass through the point intersection of two mutually perpendicular axes OX and OY.

$\Rightarrow \quad I_{zz'} = I_{xx'} + I_{yy'}$

Note :— This theorem is used only for plane lamina.

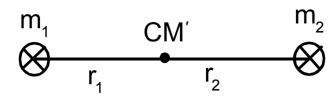
(b) Theorem of parallel axes *:-*



According to this theorem, the moment of inertia of a body about any axis is equal to the sum of moment of inertia about an axis passing through its centre of gravity and parallel to given axis and product of its mass and square of distance between the centre of gravity and the axis of rotation.

$I_{xx'} = I_{C.G} + Md^2$

(c) The moment of inertia of two point masses about their centre of mass -



Let m_1 and m_2 be two masses distant r from each other and r_1 and r_2 be the distances of their centre of mass from m_1 to m_2 respectively.

1) $r_1 + r_2 = r$

2) $m_1 r_1 = m_2 r_2$

3) $r_1 = \dfrac{m_2 r}{m_1 + m_2}$, $\quad r_2 = \dfrac{m_1 r}{m_1 + m_2}$

4) $I = m_1 r_1^2 + m_2 r_2^2$

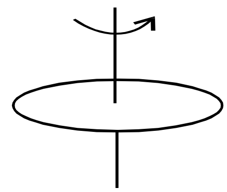
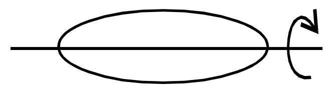
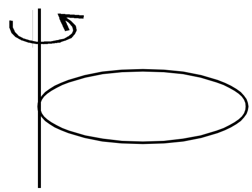
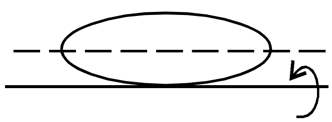
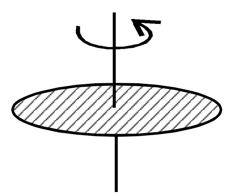
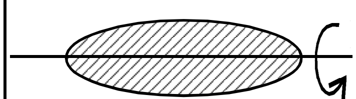
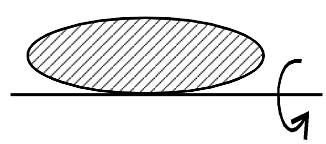
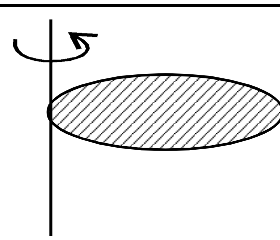
5) $I = \left(\dfrac{m_1 m_2}{m_1 + m_2}\right) r^2 = \mu r^2$

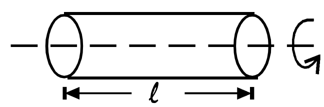
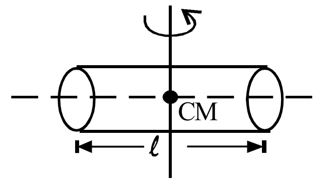
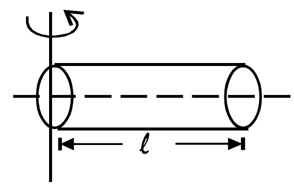
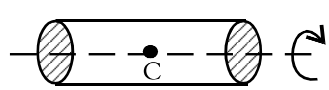
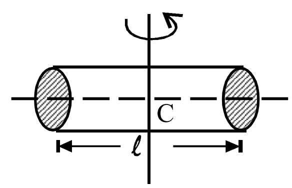
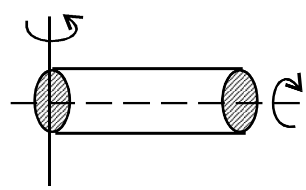
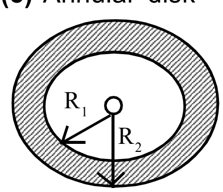
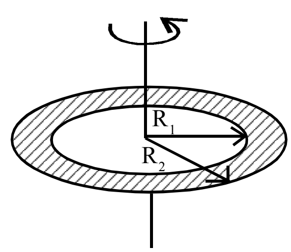
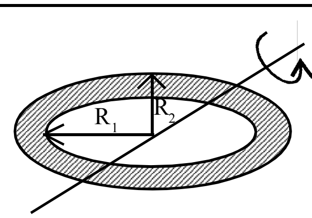
where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is known as reduced mass

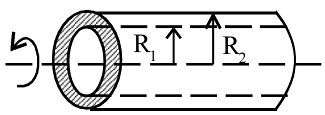
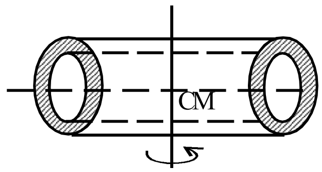
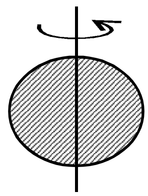
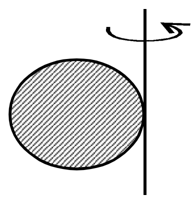
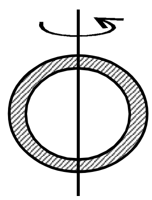
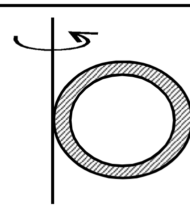
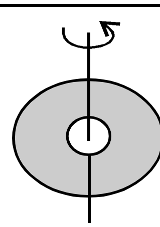
$\mu < m_1$ and $\mu < m_2$

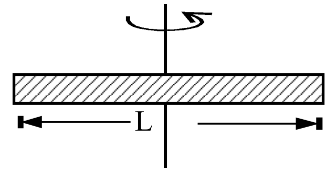
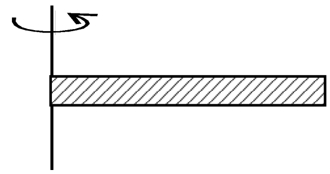
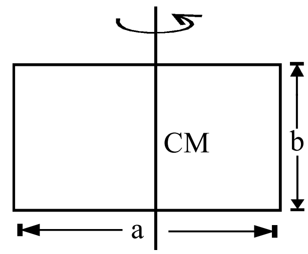
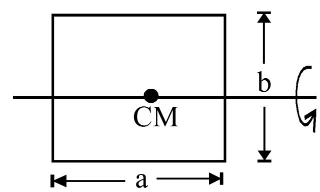
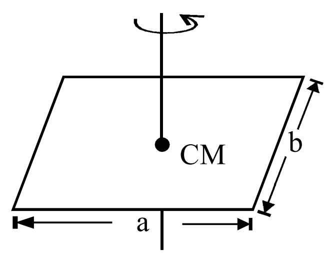
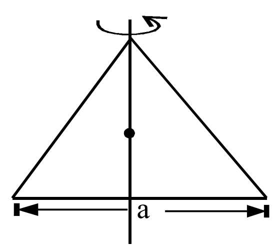
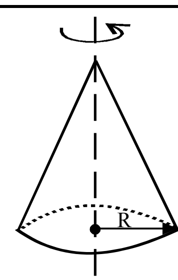
6) In Diatomic molecules like H_2, HCl etc. moment of inertia about there center of mass is derived from above formula.

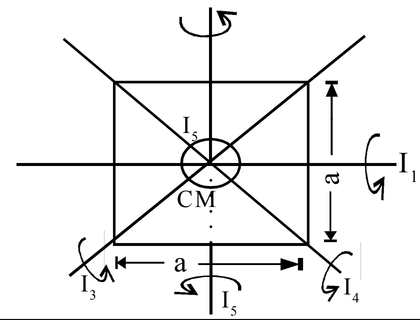
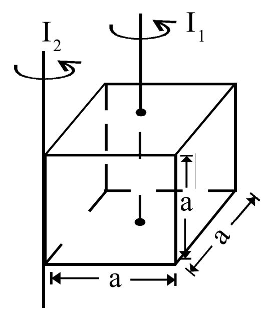
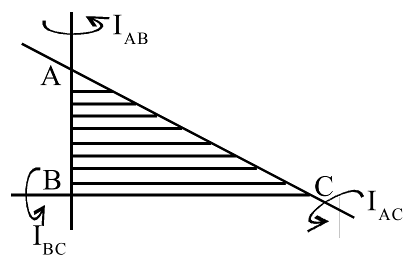
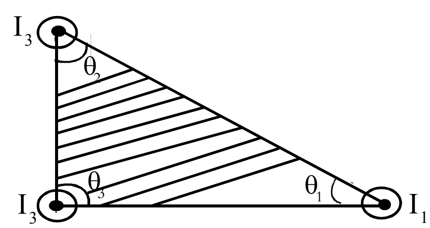
FORMULAE FOR THE MOMENT OF INERTIA OF REGULAR BODIES

Shape of body	Axis of Rotation	Figure	Moment of Inertia (I)	Radius of Gyration (K)
(1) Circular Ring M:- Mass R:- Radius	**1)** Passes through the centre and perpendicular to the plane		MR^2	R
	2) About its Diameter in its own plane		$(1/2)MR^2$	$R/\sqrt{2}$
	3) About a tangential axis perpendicular to its own plane.		$2MR^2$	$\sqrt{2}\,R$
	4) About a tangential axis in its own plane		$\frac{3}{2}\ MR^2$	$\sqrt{\frac{3}{2}}.R$
(2) Circular disc M:- Mass R:- Radius	**1)** Passing through the centre and perpendicular to the plane		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	2) About Diameter		$MR^2/4$	$\frac{R}{2}$
	3) About a tangential axis lying in its own plane.		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	4) About a tangential axis Perpendicular to its own plane		$\frac{3}{2}\ MR^2$	$\sqrt{\frac{3}{2}}.R$

3) Hollow Cylinder M = Mass R = Radius L = Length	**a)** about its geometrical axis		MR^2	R
	b) about an axis passing through its CM and perpendicular to its length		$\sqrt{\frac{R^2}{2}+\frac{\ell^2}{12}}$	
	c) about an axis perpendicular to its length and passing through one end of the cylinder		$\sqrt{\frac{R^2}{2}+\frac{\ell^2}{3}}$	
(4) Solid Cylinder M:- Mass R:- Radius L:- Length	**A)** About its geometrical Axis		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	B) About an axis passing through its C.G.and Perpendicular to its axis		$\frac{MR^2}{4}+\frac{M\ell^2}{12}$	$\sqrt{\frac{R^2}{4}+\frac{\ell^2}{12}}$
	C) About the diameter of one of the faces of the cylinder and perpendicular to the length .		M $\sqrt{\frac{R^2}{2}+\frac{\ell^2}{3}}$	
(5) Annular disk 	**A)** Passing through centre and perpendicular to the plane		$\frac{M}{2}[R_1{}^2 + R^2{}_2]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
M:- Mass, R_1 : Internal Radius R_2 : Outer Radius	**B)** About its diameter		$\frac{M[R_1^2+R_2^2]}{4}$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$

(6) Hollow Cylinder R_1 : Internal Radius R_2 :- Outer Radius M :- Mass L :- Length.	**1)** About geometrical Axis or about the Axis which is passing through centre		$\frac{M(R_1^2+R_2^2)}{2}$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	2) Passing through centre of mass and perpendicular to its length		$M\sqrt{\frac{L^2}{12}+\frac{R_1^2+R_2^2}{4}}$	
(7) Solid Sphere M:- Mass R:- Radius	**A)** About its axis OR diameter, which is passing through centre.		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}.R$
	B) About Tangential		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}.R$
(8) Thin Spherical Shell	**1)** Passing through axis or diameter.		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}.R$
(Hollow Sphere) M : Mass R : Radius Thickness negligible	**2)** About Tangential Axis		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}.R$
(9) Solid sphere with cavity Internal radius = r Outer Radius = R Mass :- M	About passing through centre OR about diameter		$\frac{2}{5}M\frac{[R^5-r^5]}{[R^3-r^3]}$	$\sqrt{\frac{2}{5}\frac{(R^5-r^5)}{(R^3-r^3)}}$

(10) Thin rod [thickness is negligible w.r.t. length]	**1)** Passing through centre of mass and perpendicular to length		$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$
	2) Passing through its one end and perpendicular to Axis.		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(11) Rectangular plate a = Length b = width	**(a)** about an axis passing through CM and perpendicular to side a in its plane		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
M = Mass	**b)** about an axis passing through CM and perpendicular to side b in its plane.		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
	c) about an axis passing through CM and		$\frac{M(a^2+b^2)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$
(12) Triangular Prism Side of base is (a) and height (a)	**1)** Passing through centre of mass and perpendicular triangular face		$\frac{Ma^2}{6}$	$\frac{a}{\sqrt{6}}$
(13) Cone Radius :- R height :- h	**1)** About the line joining of top of the cone and mid point of base		$\frac{3}{10}MR^2$	$\sqrt{\frac{3}{10}} \times R$

SPECIAL POINT			
(a)	Moment of inertia of square plate	$I_1 = I_3 = I_4 = \frac{Ma^2}{12}$ $I_5 = \frac{Ma^2}{6}$	
(b)	Momentum of inertia of cube :-	$I_1 = \frac{Ma^2}{6}$ $I_2 = \frac{2Ma^2}{3}$	
(c)	In a triangle , M.I. will be maximum relative to smallest side.	If AC > B C > AB, $I_{AC} < I_{BC} < I_{AB}$	
(d)	In triangle , M.I. will be maximum relative to that perpendicular axis which passes through least angle.	If $\theta_1 < \theta_2 < \theta_3$ $I_1 > I_2 > I_3$	
(e)	Greater the mass away from axis of rotation , more will be MI.		

Examples based on Moment of intertia

Ex.8 If the radius of solid sphere is 35 cm. The ratio of radius of gyration, when the axis is along a diameter to that when the axis is along a tangent will be-

(A) $\sqrt{\frac{10}{35}}$ (B) $\sqrt{\frac{35}{10}}$

(C) $\frac{7}{1}$ (D) $\frac{1}{7}$

Sol.(A) Along the diameter, I_g = (2/5) mR^2 or

K_g^2 = (2/5) R^2 or $K_g = R\sqrt{\frac{2}{5}} = 35\sqrt{\frac{2}{5}}$

$= 7\sqrt{10}$ cm

Along tangent, $I = Ig + mR^2$

$\therefore \quad mK^2 = mK_g^2 + mR^2$ or $K^2 = K_g^2 + R^2$

$= (2/5)\ R^2 + R^2 = (7/5)\ R^2$

$K = R\sqrt{\frac{7}{5}} = 35\sqrt{\frac{7}{5}}$, Now $\frac{K_g}{K} = \sqrt{\frac{10}{35}}$

Ex.9 The diameter of flywheel increases by 1%. The percentage increase in moment of inertia about axis of symmetry will be-

(A) 1 % (B) 2%
(C) 3% (D) 4%

Sol.(B) The moment of inertia of flywheel is given by,

$I = MR^2$

Taking log,

$\log I = \log M + 2 \log R$

Differentiating, $\frac{dI}{I} = 0 + 2\frac{dR}{R}$

$\therefore$ % change in moment of inertia

$= \frac{dI}{I} \times 100 = 2 \times 1\% = 2\%$

Ex.10 The moment of inertia of sphere is 20 kg-m^2 about the diameter. The moment of inertia about any tangent will be-

(A) 70 kg-m^2 (B) 35 kg-m^2
(C) 50 kg-m^2 (D) 20 kg-m^2

Sol.(A) According to the theorem of parallel axes, the have

$I = I_G + Ma^2 = \frac{2}{5}MR^2 + MR^2 \quad (\because a = R)$

$= \frac{7}{5}MR^2$

Given that $\frac{2}{5}MR^2 = 20$

or $MR^2 = \frac{20 \times 5}{2} = 50$,

$\therefore I = \frac{7}{5} \times 50 = 70$ kg-m^2

Ex.11 If the moment of inertia of a disc about an axis tangentially and parallel to its surface be I, what will be the moment of inertia about the axis tangential but perpendicular to the surface-

(A) $\frac{6}{5}I$ (B) $\frac{3}{4}I$

(C) $\frac{3}{2}I$ (D) $\frac{5}{4}I$

Sol.(A) According to the theorem of parallel axes, the moment of inertia of the disc about an axis tangentially and parallel to the surface is given by

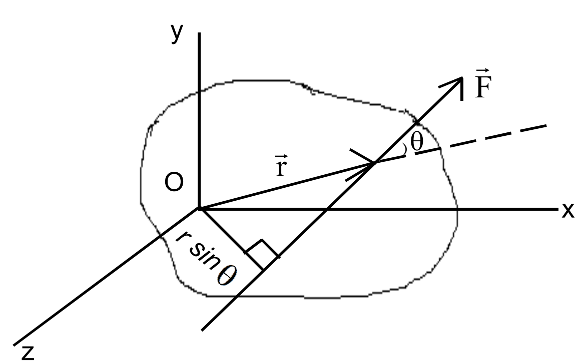
$I = I_{parallel} = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2$

The moment of inertia of the disc about an axis tangential but perpendicular to the surface is given by

$I' = I_{perpendicular} = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$

$= \frac{6}{5}\left[\frac{5}{4}MR^2\right] = \frac{6}{5}I$

5. TORQUE



(a) The torque of force F about the point O is equal to the product of force and perpendicular distance of line of action of force from point.

τ = (Force) x (Perpendicular distance of line of action of force from point O)

= Fr sin θ = (Fsin θ) r

= (The component of force perpendicular to position vector) × (Position vector)

$\therefore$ τ = Fr sin θ, r sin θ is known as lever arm

(b) **Unit :** In M.K.S = N-m

In C.G.S = dyne-cm

(c) **Dimension :** ML^2T^{-2}

(d) In vector form $\vec{\tau} = \vec{r} \times \vec{F}$

$= rF\sin\theta\,\hat{n}$, where θ is angle between $\vec{r}$ and $\vec{F}$ and $\hat{n}$ is unit vector perpendicular to the plane of $\vec{r}$ and $\vec{F}$.

(e) Torque is a vector quantity, whose direction is perpendicular to the plane of force and position vector and its direction is given by right hand screw rule.

(f) If the torque rotates the body in anticlock wise direction, the torque is positive and if the torque rotates the body in clock-wire direction, the torque will be negative.

(g) If a body is acted upon by more than one force, the total torque is the vector sum of each torque.

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_2 + ... + \vec{\tau}_n$$

(h) $\tau = I\alpha$

I - Moment of inertia with respect to axis of rotation.

α - Angular acceleration with respect to axis of rotation

τ - Torque of force which is causing the rotational motion

(i) $\vec{\tau} = \dfrac{d\vec{J}}{dt}$, where $\vec{J}$ is angular momentum

(j) The more is the value of r, the more will be torque and easier to rotate the body.

i) The handle of screw driver is taken thick.

ii) In villages the handle of flour-mill is placed near the circumference.

iii) The handle of handpump is kept-long.

iv) The rinch used for opening the tap, is kept-long.

(k) Work done by torque $= \int_{\theta_1}^{\theta_2} \tau d\theta$

= Torque × angular displacement

Examples based on Torque

Ex.12 Given that, $\vec{r} = 2\hat{i} + 3\hat{j}$ and $\vec{F} = 2\hat{i} + 6\hat{k}$. The magnitude of torque will be-

(A) $\sqrt{405}$ N.m (B) $\sqrt{410}$ N.m
(C) $\sqrt{504}$ N.m (D) $\sqrt{510}$ N.m

Sol.(C) We know that, $\vec{\tau} = \vec{r} \times \vec{F}$

$\Rightarrow \vec{\tau} = (2\hat{i} + 3\hat{j}) \times (2\hat{i} + 6\hat{k})$

$= 12(-\hat{j}) + 6(-\hat{k}) + 18\hat{i}$

$= -12\hat{j} - 6\hat{k} + 18\hat{i}$

[Note : $\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = k$, $\hat{j} \times \hat{i} = -k$ etc]

Now, $|\vec{\tau}| = \sqrt{(-12)^2 + (-6)^2 + (18)^2}$

$= \sqrt{144 + 36 + 324} = \sqrt{504}$

Ex.13 A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 seconds. The magnitude of this toque is-

(A) $3A_0/4$ (B) A_0
(C) $4A_0$ (D) $12A_0$

Sol.(A) $\Delta J = 4A_0 - A_0 = 3A_0$ and $\Delta T = 4$

$\therefore \tau = \dfrac{\Delta J}{\Delta T} = \dfrac{3A_0}{4}$

Ex.14 The moment of inertia of a wheel is 1000 kg-m^2. At a given instant, its angular velocity is 10 rad/s. After the wheel rotates through an angle of 100 radians the wheel's angular velocity is 100 rad/s. The torque applied on wheel is-

(A) 4.95×10^5 N.m (B) 4.95×10^4 N.m

(C) 4.95×10^3 N.m (D) 49.5×10^5 N.m

Sol.(B) We know that $\omega^2 - \omega_0^2 = 2\,\alpha\,\theta$

$$\therefore \quad \alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{(100)^2 - (10)^2}{2 \times 100}$$

$$= 49.5 \text{ rad/s}^2$$

$$\tau = I\,\alpha = 1000 \times 49.5 = 4.95 \times 10^4 \text{ N-m.}$$

Ex.15 An automobile engine develops 100 kilo-watt, when rotating at a speed of 1800 rev/min. The torque developed by it will be-

(A) 60 N-m (B) 531 N-m

(C) 5.31 N-m (D) 6.0 N-m

Sol.(B) The power delivered by th torque τ exerted on rotating body is given by

$$P = \tau\omega \text{ or } \tau = \frac{P}{\omega}$$

Here P = 100 KW = 100, 000 Watt

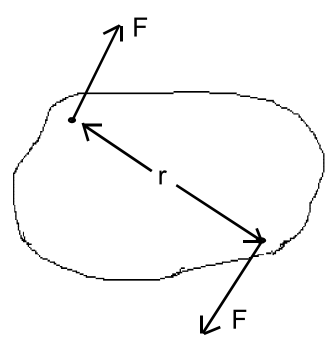
$$\omega = \left(\frac{1800}{60}\right) \times 2\pi$$

$$= 60\,\pi \text{ rad/sec,}$$

$$\tau = \frac{10^5}{60 \times 3.14}$$

$$= 531 \text{ N-m}$$

6. FORCES COUPLE



(a) When two forces of equal magnitude act on different points and in opposite direction, these force form a couple.

(b) The effect of couple is known by its moment.

(c) The moment of couple is equal to the product of magnitude of any force and perpendicular distance between the force.

Moment of couple = (F) (r)

(d) The couple causes the rotational motion in the body.

(e) Generally both couple and torque carry equal meanings.

When there is only one force applying on a body the reaction force forms torque with it.

(f) The work done by couple on a body is equal to work done by torque

$\therefore$ Work done by couple = work done by torque

$$= \int \tau\, d\theta$$

(g) If particle complets n rotations under the effect of couple or torque, work done

$$W = \tau\,(2\pi\, n)$$

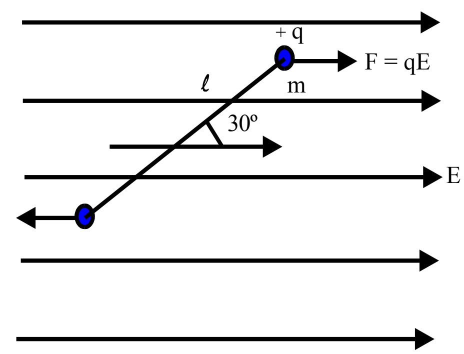
(h) As on stretching a spring energy is stored in it, in the same way as on twisting a wire the work done by torque is stored in the form of energy.

$$W = \int_0^\theta C\theta\, d\theta = \frac{1}{2}\, C\theta^2$$

$\tau = C\,\theta$ [Restoring torque],
(where C = twisting coefficient)

Example based on **Couple**

Ex.16 Two particles having charges +q and –q are connected to two ends of a rod of length ℓ and rod making an angle of 30º with the electric field direction then the couple acting is



(A) $q\,l\,E$ (B) $q\,l\,E/2$

(C) qE (D) $2\,q\,l\,E$

Sol. Couple =

Force on one charge x Perpendicular distance between charges

$\tau = qE \times l \sin 30^\circ$

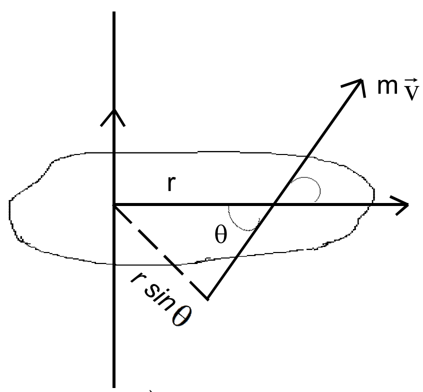
$= \frac{qEl}{2}$

7. ANGULAR MOMENTUM

(a) The moment of linear momentum of a body with respect to any axis of rotation is known as angular momentum.

(b) It is a vector quantity, which is often represented by $\vec{L}$ or $\vec{J}$

(c) Angular momentum $\vec{J} = \vec{r} \times \vec{P}$



$= \vec{r} \times (m\vec{v})$

$= m(\vec{r} \times \vec{v})$

or $\vec{J} = rp \sin\theta\,\hat{n}$

$= mvr \sin\theta\,\hat{n}$

θ is angle between $\vec{r}$ and $\vec{v}$ $\hat{n}$,

is unit vector perpendicular to the plane of $\vec{r}$ and $\vec{v}$

(d) The direction of angular momentum is perpendicular to the plane of $\vec{r}$ and $\vec{v}$ and it is given by right hand screw rule.

(e) $J = mvr \sin\theta$

Cases :

(I) If $\theta = 0$, $J = 0$ (Minimum)

(II) If $\theta = 90^\circ$, $J = mvr$ (Maximum)

$= (mr^2)\,\omega$: $\because v = r\omega$.

(f) Unit : J. second , kg m^2/s, kg m^2 rad/sec

(g) Dimension : $[M^1L^2T^{-1}]$

(h) If direction of rotation is anticlockwise, angular momentum is taken positive and if direction of rotation is clockwise, angular momentum is taken negative.

(i) The angular momentum of a system of particle's is equal to the vector sum of angular momentum of each particle.

$$\vec{J} = \vec{J_1} + \vec{J_2} + \vec{J_3} + \ldots\ldots\ldots\ldots$$

(j) Relation between angular momentum and angular velocity

$$J = I\,\omega$$

I - Moment of inertia with respect to axis of rotation

ω - Angular velocity due to angular momentum

J - The moment of momentum which is causing rotational motion.

(k) The rate of change of angular momentum is equal to the torque applied on the body.

$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

(l) In rotational motion angular momentum has equal importance as linear momentum in linear motion

(m) If torque acting of a particle is zero then

$$\vec{\tau} = 0 \Rightarrow \frac{d\vec{J}}{dt} = 0$$

Which implies that the angular momentum remains conserved when no external force acts on the body

7.1 Law of Conservation of Angular Momentum

(a) If no external torque is acting upon a body rotating about an axis, then the angular momentum of the body remains constant that is,

$$J = I\omega = \text{constant}$$

(b) **Proof :–** We have read above that when a body rotates about an axis under the action of an external torque τ, the rate of change of angular momentum of the body is equal to the torque; that is,

$$\frac{dJ}{dt} = \tau$$

If the external torque is zero ($\tau = 0$), $\frac{dJ}{dt} = 0$

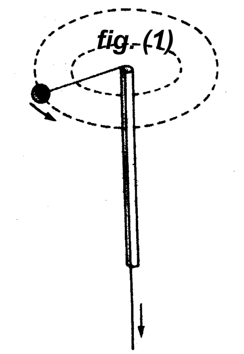
$\Rightarrow$ J = constant

This is the law of conservation of angular momentum.

Examples :-

fig. (2)

fig. (1)



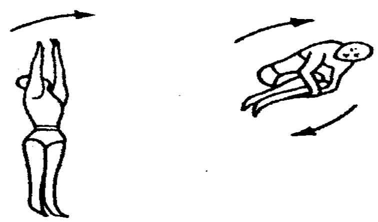
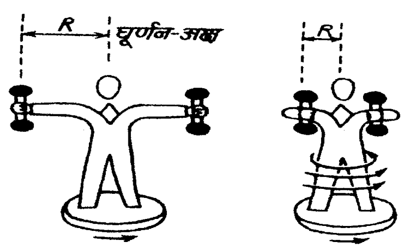


fig. (3)



(i) Suppose a ball is tied at one end of a cord whose other end passes through a vertical hollow tube. The tube is held in one hand and the cord in the other. The ball is set into rotation in a horizontal circle. If the cord is pulled down, shortening the radius of the circular path of the ball, the ball rotates faster than before. The reason is that by shortening the radius of the circle, the moment of inertia of the ball about the axis of rotation decreases. Hence, by the law of conservation of angular momentum, the angular velocity of the ball about the axis of rotation increases. [fig. (1)]

(ii) When a diver jumps into water from a height, he does not keep his body straight but pulls in his arms and legs towards the centre of his body. On doing so, the moment of inertia I of his body deceases. But since the angular momentum I ω reamains constant, his angular velocity ω correspondingly increases. Hence during jumping he can rotate his body in the air. [fig. (2)]

(iii) In a man with his arms outstretched and holding heavy dumb bells in each hand, is standing at the centre of a rotating table. When the man pulls in his arms, the speed of rotation of the table increases. The reason is that on pulling in the arms, the distance R of the dumbells from the axis of rotation decreases and so the moment of inertia o the man decreases. Therefore, by conservation of angular momentum, the angular velocity increases. [fig. (3)]

In the same way, the ice skater and the ballet dancer increase or decrease the angular velocity of spin about a vertical axis by pulling or extending out their limbs.

Examples based on Angular momentum

Ex.17 A thin circular ring of mass M and radius r is rotating about an axis passing through its centre and perpendicular to its plane with a constant angular velocity. Two objects, each of mass m are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with angular velocity -

(A) $\frac{\omega(M-2m)}{M+2m}$ (B) $\omega M\ (M - m)$

(C) $\frac{\omega(M+2m)}{M}$ (D) $\frac{\omega M}{M+2m}$

Sol.(D) As there is no external force acting an the system, so angular momentum will remain conserved.

Now $J = I_1\ \omega_1 = I_2\omega_2$

$$MR^2\omega = (MR^2 + mR^2 + mR^2)\ \omega_2$$

$$\Rightarrow \quad \omega_2 = \frac{M\omega}{M+2m}$$

Ex.18 In a play ground there is a merry go round of mass 120 kg and radius 4 m. The radius of gyration is 3 m. A child of mass 30 kg runs at a speed of 5 m/sec tangent to the rim of the merry go round when it is at rest and then jumps on it. It we neglect the friction, the angular velocity of the merry-go-ground and child will be-

(A) 0.1 rad/sec (B) 0.2 rad/sec

(C) 0.4 rad/sec (D) 0.8 rad/sec

Sol.(C) $m_c vr = I\omega = [m_c r^2 + mk^2]\omega$

Given that, r = 4 m, m = 120 kg

$\therefore \quad 30 \times 5 \times 4 = (120 \times 3^2 + 30 \times 4^2)\ \omega$

and m_c = 30 kg

or $\omega = \dfrac{600}{1080+480}$ = 0.4 rad/sec

Ex.19 A body of mass 1.0 kg is rotating on a circular path of diameter 2.0 m at the rate of 10 rotations in 31.4 sec. The angular momentum of the body is- (in kg. m^2/s)

(A) 3 (B) 4 (C) 2 (D) 1

Sol.(C) Mass of the body, m = 1.0 kg

The distance of the body from the axis of rotation, $r = \dfrac{2.0}{2}$ = 1.0 m

$\therefore$ Moment of inertia of the body about the axis of rotation is ,

$I = mr^2 = (A).(A)^2 = 1$ kg-m^2

Angular velocity of the body, $\omega = 2\pi n$, where n is the number of rotations per/sec

Here, $n = \dfrac{10}{31.4}$

$\therefore \quad \omega = 2 \times 3.14 \times \dfrac{10}{31.4}$ = 2 rad/s

$\therefore$ Angular momentum, J = $I\omega$ = 1 × 2

= 2 kg-m^2/s

Ex.20 The angular momentum of the earth rotating about its own axis will be- (Mass of earth = 5.98 × 10^{27} gm and mean radius R of earth = 9.37 x 10^6 m)

Sol. Given, M = 5.98 × 10^{27} gm = 5.98 × 10^{24} kg and R = 9.37 × 10^6 m

Angular velocity, $\omega = \dfrac{2\pi \text{ radian}}{1 \text{ day}}$

$= \dfrac{2\pi}{24 \times 60 \times 60}$ rad/sec

Moment of inertia of earth

$I = \dfrac{2}{5} MR^2 = \dfrac{2}{5} \times (5.98 \times 10^{24})\ (9.37 \times 10^6)^2$

$\therefore$ J = $I\omega$ = 1.53 × 10^{34} kg m^2/sec

(Putting the values of ω and I)

Ex.21 The z-component of angular momentum in terms of linear momentum will be-

(A) $J_z = xp_y - yp_x$ (B) $J_z = yp_y - xp_x$

(C) $J_z = zp_y - yp_z$ (D) $J_z = zp_x - xp_z$

Sol.(A) $J_z = xp_y - yp_x$, because $\vec{J} = \vec{r} \times \vec{p}$

8. KINETIC ENERGY OF ROTATION

(a) The energy due to rotational motion of a body is known as rotational kinetic energy.

(b) If I be moment and inertia of body about axis of rotation and ω be its angular velocity, then kinetic energy of rotation.

$$E_r = \frac{1}{2} I\,\omega^2 \quad \text{or} \quad E_r = \frac{1}{2} MK^2\,\omega^2$$

$$= \frac{1}{2}\frac{J^2}{I} = \frac{1}{2}\frac{\vec{J}\cdot\vec{J}}{I}$$

(c) If ω is constant, $E_r \propto I$

(d) If I is constant, $E_r \propto \omega^2$

(e) If Jis constant, $E_r \propto \dfrac{1}{I}$

(f) **Work energy theorem :-**

The work done by torque = change in kinetic energy of rotation

(g) Power of rotation $P = \dfrac{dE_r}{dt} = \tau\,\omega = \left(\dfrac{d\vec{J}}{dt}\right) \cdot \left(\dfrac{\vec{J}}{I}\right)$

(h) If a body performs rotational kinetic motion as well as linear motion, then the total energy of body is equal to the sum of kinetic energy of rotational motion and kinetic energy of linear motion.

$\therefore$ Total K.E = $E_r + E_t$

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2$$

(i) If rotation is taking place without skidding then $v = r\omega$, $a = r\alpha$

(j) Unit : same as that of energy

(k) Dimension : $M^1L^2T^{-2}$

Examples based on Kinetic energy of rotation

Ex.22 The moments of inertia of two rotating bodies A and B are I_A and I_B ($I_A > I_B$) and their angular momentum are equal. If their kinetic energies be K_A and K_B, respectively, then-

(A) $\frac{K_A}{K_B} > \frac{1}{1}$ (B) $\frac{K_B}{K_A} > \frac{1}{1}$

(C) $\frac{K_A}{K_B} = 1$ (D) $\frac{K_A}{K_B} = \frac{1}{2}$

Sol.(B) The kinetic energy of a rotating body is, $K = 1/2\, I\omega^2$

and the angular momentum is, $J = I\omega$

$$\therefore \quad K = \frac{J^2}{2I}$$

Let K_A and K_B be the kinetic energies of A and B. The angular momentum of each is J. Then

$$\frac{K_A}{K_B} = \frac{J^2/2I_A}{J^2/2I_B} = \frac{I_B}{I_A}$$

But $I_A > I_B$ (given), $\therefore K_B > K_A$

Ex.23 The moment of inertia of a body about a given axis 1.2 kg-m^2. Initially, the body is at rest. In order to produce a rotational K.E. of 1500 joule, an angular acceleration of 25 rad/s^2 must be applied about that axis for a duration of-

(A) 4 s (B) 2 s

(C) 8 s (D) 10 s

Sol.(B)

$$K.E. = (1/2)\, I\omega^2$$

$$= (1/2)\, I\, (\alpha t)^2 = (1/2)\, I\alpha^2 t^2$$

$\therefore$ $1500 = (1/2)\,(1.2)\,(25)^2\, t^2$ or $t = 2$ s

Ex.24 If a flywheel of mass 20 kg and diameter 1 m is rotating 300 revolutions per minute, its kinetic energy will be -

(A) 2465 J (B) 2.465 J

(C) 24.65 J (D) 246.5 J

Sol.(A) Here, $\omega = \frac{300 \times 2\pi}{60} = 31.4$ rad/sec

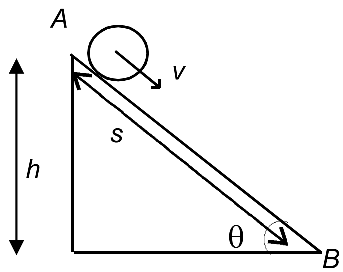
$$I = mR^2 = 20\,(1/2)^2$$

$$= 5 \text{ kg m}^2$$

$$\therefore K.E. = \frac{1}{2} I\omega^2 = \frac{1}{2} \times 5 \times (31.4)^2 = 2465 \text{ J}$$

9. LINEAR AND ROLLING MOTION OF A BODY ON INCLINED PLANE

Linear Motion of Body on Inclined Plane -



Let the length of inclined plane is s and its inclination from horizontal be θ

(a) Linear acceleration of body $a_{linear} = g \sin\theta$

(b) Angular acceleration = Zero

(c) The velocity acquired by body on reaching the lowest point B is $v_{linear} = \sqrt{2gh}$

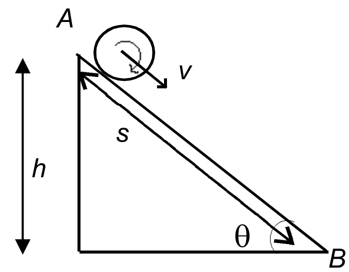
(d) Angular velocity = Zero

(e) Time taken by body to reach lowest point B is $t_{linear} = \sqrt{\frac{2s}{a_{linear}}} = \sqrt{\frac{2s}{g\sin\theta}}$

(f) The loss in potential energy on reaching the point B = Increase in Kinetic energy

$= mgh = \frac{1}{2}mv^2$

Rolling Motion of a Body on Inclined Plane-



Let the length of inclined plane is s and its inclination from horizontal be θ

(a) Decrease in potential energy = Increase in kinetic energy $mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$

(b) Because the motion of body is without sliding so $v = R\omega$ and $a = R\alpha$

(c) Velocity on reaching lowest point is where,

$$= \sqrt{\frac{2gh}{1+K^2/R^2}}$$

K is radius of gyration and R is the radius of body

(d) Linear accleration on reaching the lowest point

$$a = \frac{g\sin\theta}{1+K^2/R^2}$$

(e) Time taken to reach the lowest point of the plane is $t = \sqrt{\frac{2s(1+K^2/R^2)}{g\sin\theta}}$

(f) Angular accleration $\alpha = \frac{a}{R}$,

angular velocity $= \frac{v}{R}$

Examples based on **Linear and rolling motion of a body on inclined plane**

Ex. 25 A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotation is-

(A) 2/5 (B) 3/5 (C) 2/7 (D) 3/7

Sol.(C) Total energy,

$E = (1/2)\ I\omega^2 + (1/2)\ mv^2$

$= (1/2)\ (2/5\ mr^2)\ \omega^2 + (1/2)\ mr^2\omega^2$

$= (1/5)\ mr^2\omega^2 + (1/2)\ mr^2\omega^2 = (7/10)\ mr^2\omega^2$

Rotational energy $= (1/5)\ mr^2\omega^2$

$$\therefore \frac{\text{Rotational energy}}{\text{Total energy}} = \frac{\frac{1}{5}mr^2\omega^2}{\frac{7}{10}mr^2\omega^2} = \frac{2}{7}$$

Ex.26 A solid sphere and a solid cylinder having the same mass and radius, roll down the same incline. The ratio of their acceleration will be -

(A) 15 : 14 (B) 14 : 15

(C) 5 : 3 (D) 3 : 5

Sol.(A)

We know that, $a = \dfrac{g\sin\theta}{(1+k^2/R^2)}$

Here, $a_1 = \dfrac{5g\sin\theta}{7}$ and $a_2 = \dfrac{2g\sin\theta}{3}$, $\therefore$

$a_1 : a_2 = 15 : 14$

Ex.27 A sphere is rolling down without slipping in the incline plane from a vertical height h. The linear velocity as it reaches the ground, if its mass is m and radius is r, will be-(k is radius of gyration of sphere)

(A) $\sqrt{\dfrac{2gh}{1+2k^2/r^2}}$ (B) $\sqrt{\dfrac{2gh}{1+k^2/2r^2}}$

(C) $\sqrt{\dfrac{2gh}{1+k^2/r^2}}$ (D) $\sqrt{\dfrac{gh}{1+k^2/r^2}}$

Sol.(C)

When the sphere reaches the ground, its P.E. is converted into its. K.E.

$\therefore$ K.E. $= (1/2)\ mv^2 + (1/2)\ I\omega^2$

$= (1/2)\ mv^2 + (1/2)\ (mk^2)\ \omega^2$

$mgh = (1/2)\ m\ (v^2 + k^2\omega^2)$

or $2gh = \omega^2\ (r^2 + k^2)$ ($\because v = r\omega$)

$$\therefore \omega = \sqrt{\frac{2gh}{r^2+k^2}}, \quad v = \sqrt{\frac{2gh}{1+k^2/r^2}}$$

Ex.28 A body of mass m slides down an incline and reaches the bottom with a velocity v. If the same mass were in the form of a ring, which rolls down this incline, the velocity of the ring at bottom would have been-

(A) v (B) $\sqrt{2}$ v

(C) $\dfrac{1}{\sqrt{2}}$ v (D) $\sqrt{2/5}$ v

Sol.(C) For slidding, a = g sin θ. Hence the velocity v is given by,

$v^2 = 0 + 2\ (g \sin\theta) \times \ell$ ....(i)

For rolling, acceleration down the inclined plane is given by,

$$a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)} = \frac{1}{2}\ g \sin\theta\ (\because k^2 = R^2)$$

In case of ring,

$$V_r^2 = 2 \times (1/2\ g \sin\theta) \times \ell = \frac{v^2}{2},$$

$$\therefore V_r = \frac{v}{\sqrt{2}}$$

Ex.29 When a sphere rolls without slipping, the ratio of its kinetic energy of translation to its total kinetic energy is-

(A) 1: 7 (B) 1 : 2 (C) 1 : 1 (D) 5 : 7

Sol.(D) $$\frac{E_{trans}}{E_{total}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mv^2+\frac{1}{2}I\omega^2}$$

$$= \frac{\frac{1}{2}mv^2}{\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{2}{5}mr^2\right)x\frac{v^2}{r^2}}$$

$$= \frac{1}{1+(2/5)} = \frac{5}{7}$$

$[\because I = \frac{2}{5}\ mr^2,\ v = r\omega]$

POINTS TO REMEMBER

(a) A rigid body is said to be in general motion, if it has both the translational and rotational motions.

(b) Moment of force about the axis of rotation is called torque.

(c) Torque = Force x perpendicular distance of point of application of force from the axis of rotation.

$\tau = r.\ F$ or $\vec{\tau} = \vec{r} \times \vec{F}$ or $\tau = rF\ \sin\theta$

where θ is the angle between $\vec{r}$ and $\vec{F}$.

(d) Torque may be clockwise or anticlockwise. Anticlockwise torque is taken as positive.

(e) Power of Torque $P = \tau \times \omega$ where $\omega = \frac{d\theta}{dt}$

(f) **Theorem of parallel axis** :— This theorem states that the moment of inertia of a body about an axis parallel to the axis passing through C.G. is equal to the sum of its moment of inertia about C.G. and Ma^2, where M is the total mass of the body and 'a' is the perpendicular distance between two parallel axes. Thus

$$I = I_{CM} + Ma^2$$

Where,

I_{CM} = Moment of inertia of the body about centre of mass,

M = Total mass of the body,

a = perpendicular distance between two parallel axes.

(g) **Theorem of perpendicular axes** :— This theorem states that the moment of inertia of the body about an axis perpendicular to the plane of the body and passing through the point of intersection of two mutually perpendicular axes (lying in the plane of body) is equal to the sum of its moment of Inertias about those axes i.e.

$I_z = I_x + I_y$ where I_x = MI about X axis

I_y = MI about Y axis

(h) **Kinetic energy of rotation** :—

$$K_{rot} = \frac{1}{2} I\omega^2$$

(i) **Total energy of a rolling body** :—

$$K = K_{trans} + K_{rot} = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

(j) **Acceleration of a body rolling down an inclined plane.** :—

$$a = g\ \sin\theta \left(1 + \frac{1}{mR^2}\right)$$

SOLVED EXAMPLES

Ex.1 The radius of a wheel of a car is 0.4 meter. The car is accelerated from rest by an angular acceleration of 1.5 rad/sec^2 for 20 seconds. Distance covered by wheel and linear velocity will respectively be -

(A) 120 m,12 m/s (B) 12 m,12 m/s
(C) 1.2 m,12 m/s (D) 120 m,1.2 m/s

Sol.(A) Initially the wheel is at rest ($\omega_0 = 0$). The angular displacement of the wheel in t sec is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 0 + \frac{1}{2} (1.5 \text{ rad/sec}^2)(20 \text{ sec})^2 = 300 \text{ radian}$$

Radius of the wheel, r = 0.4 meter. Therefore, the linear displacement of the wheel is -

$$s = r\theta$$

$$= 0.4 \text{ meter} \times 300 \text{ rad} = 120 \text{ meter}$$

This is the distance covered by the wheel.

Angular velocity of the wheel after t sec is

$$\omega = \omega_0 + \alpha t$$

$$= 0 + (1.5 \text{ rad/sec}^2)(20 \text{ sec}) = 30 \text{ rad/sec}$$

Therefore, the linear velocity of the wheel is

$$v = r\omega$$

$$= 0.4 \text{ meter} \times 30 \text{ rad/sec} = 12 \text{ meter/sec}$$

Ex.2 A wheel of mass 6 kg is rotating at 300 rpm. Its angular velocity will be

(A) 31.4 rad/sec (B) 3.14 rad/sec
(C) 0.314 rad/sec (D) 0.03 rad/sec

Sol.(A) Here , $\omega = \frac{2\pi n}{t} = \frac{2\times3.14\times300}{60}$

$$= 31.4 \text{ rad/sec}$$

Ex.3 The shaft of an eletric motor starts from rest and on the application of a torque, it gains an angular acceleration given by $\alpha = 3t - t^2$, during the first 2 seconds after it starts after which $\alpha = 0$. The angular velocity after 6 seconds will be-

(A) 10/3 rad/sec (B) 20/3 rad/sec
(C) 5/3 rad/sec (D) 1/3 rad/sec

Sol.(A) Given, $\alpha = 3t - t^2 \Rightarrow \frac{d\omega}{dt} = 3t - t^2$ or

$$\therefore d\omega = (3t - t^2)\, dt \Rightarrow \int d\omega = \int (3t - t^2)\, dt$$

$$\Rightarrow \omega = \left(\frac{3t^2}{2} - \frac{t^3}{3}\right) + C$$

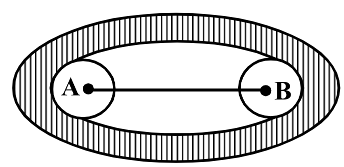
[t = 0, ω = 0 $\therefore$ c = 0]

$$\Rightarrow \omega = \frac{3t^2}{2} - \frac{t^3}{3} \quad \text{Putting } t = 2$$

$$= 6 - \frac{8}{3} = \frac{10}{3} \text{ rad/sec}$$

Since there is no angular acceleration after 2 sec the angular velocity after 6 sec remains the same i.e. 10/3 rad/sec.

Ex.4 A wheel 'A' has a radius 20 cm coupled by belt to wheel B of radius 30 cm as shown in fig. Wheel A increases its angular speed from rest at a uniform rate of 3.14 rad/sec^2. The time for wheel B to reach a rotational speed of 100 rev/min assuming that belt does not slip, will be-



(A) 5 sec (B) 10 sec
(C) 2.5 sec (D) 20 sec

Sol.(A) As the belt does not slip, velocity of A = velocity B

i.e. $v_A = v_B$ or $r_A\omega_A = r_B\omega_B$

Given, r_A = 20 cm, r_B = 30 cm

and $\omega_B = 2\pi \times 100/60$ rad/sec

So, $20\,\omega_A = 30 \times 2\pi \times 100/60$

$= 100\,\pi$

or $\omega_A = 5\pi$ rad/sec

We know that, $\omega = \omega_0 + \alpha t$ or

$t = \frac{\omega}{\alpha}$ (as $\omega_0 = 0$)

$$\therefore t = \frac{5\pi}{3.14} = 5 \text{ sec}$$

Ex.5 A turn table is rotating in a horizontal plane about the vertical axis passing through its centre with an angular velocity 20 rad/sec. It carries upon it a flywheel rotating with an angular velocity 40 rad/sec about a horizontal axle mounted in bearings. Find the angular velocity of the wheel as seen by an observer in the room-

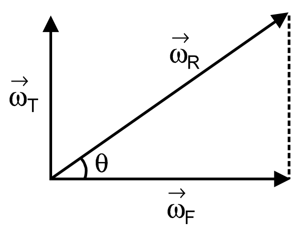
(A) 20 $\sqrt{5}$ rad/sec at an angle $\tan^{-1}$ (1/2) to the horizontal

(B) 10 $\sqrt{5}$ rad/sec at an angle $\tan^{-1}$ (1/3) to the horizontal

(C) 5 $\sqrt{5}$ rad/sec at an angle $\tan^{-1}$ (1/2) to the horizontal

(D) 20 $\sqrt{5}$ rad/sec at an angle $\tan^{-1}$ (1/6) to the horizontal

Sol.(A)



As the axis of the turn table is vertical its angular velocity ω_T is directed vertical. The axis of flywheel is horizontal therefore its angular velocity ω_F is directed horizontal, hence the resultant angular velocity is

$$\vec{\omega}_R = \vec{\omega}_F + \vec{\omega}_T$$

$$\omega_R = \sqrt{\omega_F^2 + \omega_T^2} = \sqrt{40^2 + 20^2}$$

$$= \sqrt{2000} = 20\sqrt{5} \text{ rad/sec.}$$

$\vec{\omega}_R$ lies in a plane which makes an angle θ with the horizontal

plane, given by $\theta = \tan^{-1}\left(\frac{\omega_T}{\omega_F}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Ex.6 A mass of 2 kg is rotating on a circular path of radius 0.8 m with angular velocity of 44 rad/s. If the radius of the path becomes 1.0 m, what will be the value of angular velocity ?

(A) 2.816 rad/sec (B) 3.832 rad/sec
(C) 5.899 rad/sec (D) 28.16 rad/sec

Sol.(D) Let I_1 and ω_1 be the initial moment of inertia and angular velocity of the mass about the axis of ratation and I_2 and ω_2 the corresponding quantities after the radius of the path is changed. By conservation of angular momentum, we have

$$I_1\omega_1 = I_2\omega_2$$

Here $I_1 = 2 \times (0.8)^2 = 1.28$ kg-m^2 ,

$\omega_1 = 44$ rad/s $I_2 = 2 \times (1.0)^2 = 2$ kg-m^2 ,

$\omega_2 = ?$

$\therefore\ 1.28 \times 44 = 2 \times \omega_2$

or $\omega_2 = \frac{1.28 \times 44}{2} = 28.16$ rad/s

Ex.7 A weightless horizontal rod is free to rotate about an axis OO'. Two masses, each of 1 kg, are placed at A and A' such that O'A = O'A' = 0.20 meter. Now a torque of 2.0 Newton -meter is applid on the system which rotates about OO'. If the masses are displaced to B and B' such that O'B = O'B' = 0.50 meter, then decreament in angular acceleration will be -

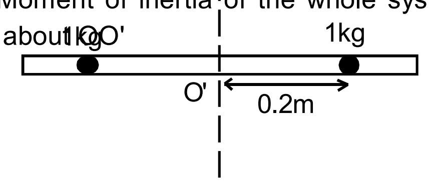
(A) 21 rad/sec^2 (B) 42 rad/sec^2
(C) 12 rad/sec^2 (D) 24 rad/sec^2

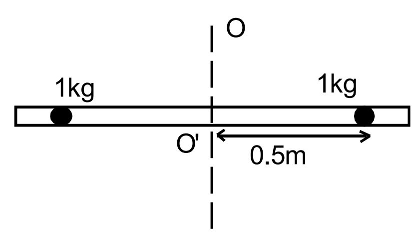
Sol.(A) In the first case, the moment of inertia of each mass about OO'

= mass × (distance from O')^2

= 1 kg × (0.20 m)^2 = 0.04 kg-m^2

$\therefore$ Moment of inertia of the whole system about OO'





(rod is weightless) is

$I = 2 \times 0.04 = 0.08$ kg-m^2

Let α be the angular acceleration

Then the torque is, $\tau = I\alpha$

$$\Rightarrow \alpha = \frac{\tau}{I} = \frac{2.0 N-m}{0.08 kg-m^2} = 25 \text{ rad/sec}^2$$

In the second case, the moment of inertia of each mass about OO'

$= 1 \text{ kg} \times (0.50 \text{ m})^2 = 0.25$ kg-m^2

$\therefore$ Moment of inertia of the whole system is

$I = 2 \times 0.25 = 0.50$ kg-m ^{2}

$\therefore$ Angular acceleration $\alpha = \frac{\tau}{I} = \frac{2.0}{0.50}$

$= 4.0$ rad/sec^2 $= 25 - 4 = 21$ rad/sec^2

Ex.8 Two spheres each of mass M and radius R/2 are connected with a massless rod of length 2 R as shown in fig. What will be the moment of inertia of the system about an axis passing through the centre of one of the sphere and perpendicular to the rod-

(A) $\frac{21}{5}$ MR^2 (B) $\frac{2}{5}$ MR^2

(C) $\frac{5}{2}$ MR^2 (D) $\frac{5}{21}$ MR^2

Sol.(A) $I = \frac{2}{5} M \left(\frac{R}{2}\right)^2 + M (2R)^2 + \frac{2}{5} M \left(\frac{R}{2}\right)^2$

$= \frac{21}{5}$ MR^2

Ex.9 Two circular discs A and B of equal masses and thickness but made of metals with densities d_A and d_B ($d_A > d_B$). If their moments of inertia about an axis passing through the centre and normal to the circular faces be I_A and I_B, then-

(A) $I_A = I_B$ (B) $I_A > I_B$

(C) $I_A < I_B$ (D) $I_A \geq I_B$

Sol.(C) $I_A = \frac{m_A r_A^2}{2}$ and

$I_B = \frac{m_B r_B^2}{2}$,

$\therefore \frac{I_A}{I_B} = \frac{r_A^2}{r_B^2}$

($\because m_A = m_B$) ....(1)

Now, $m_A = \pi r_A^2 t d_A$

$m_B = \pi r_B^2 t d_B$

So, $\pi r_A^2 t d_A = \pi r_B^2 t d_B$

or $\frac{r_A^2}{r_B^2} = \frac{d_B}{d_A}$ ....(2)

From equations (1) and (2)

$\frac{I_A}{I_B} = \frac{d_B}{d_A}$. As $d_A > d_B$ hence $I_A < I_B$

Ex.10 The moment of inertia of HCl molecule about an axis passing through its centre of mass and perpendicular to the bond will be-

(Given, internuclear distance = 1.3 Å, atomic weight of chlorine = 35 and mass of proton = 1.7×10^{-27} kg)

(A) 2.79×10^{-47} kg-m^2

(B) 27.9×10^{-47} kg-m^2

(C) 27.9×10^{-50} kg-m^2

(D) 2.79×10^{-50} kg-m^2

Sol.(A) Moment of inertia about centre of mass,

$I = \mu r^2$

μ = reduced mass $= \frac{m_H . m_{Cl}}{m_H + m_{Cl}}$

$$= \frac{1.7 \times 10^{-27} \times 35 \times 1.7 \times 10^{-27}}{1.7 \times 10^{-27} + 35 \times 1.7 \times 10^{-27}}$$

$= 1.65 \times 10^{-27}$ kg

$\therefore I = (1.65 \times 10^{-27}) (1.3 \times 10^{-10})^2$

$= 2.79 \times 10^{-47}$ kg-m^2

Ex.11 A mass m hangs from the rim of a wheel of radius r when released from rest, the mass falls through a height h in t seconds. The moment of inertia of the wheel will be-

(A) $\frac{m(g-2h)}{2h}\left(\frac{r^2}{t^2}\right)$

(B) $\frac{mr^2(g-2h)t}{2h}$

(C) $\frac{m(gt^2 - 2h)r^2}{2h}$

(D) $\frac{m(gt^2 - 2h)t^2}{2hr^2}$

Sol.(C) $h = 1/2\ at^2$ or $a = \frac{2h}{t^2} \Rightarrow \alpha = \frac{a}{r} = \frac{2h}{rt^2}$

Now $mg - T = ma$

$\Rightarrow T = m(g - a)$

$\tau = I\alpha$ or $Tr = I\alpha$ or $m(g - a) . r = I\alpha$

$$\Rightarrow \quad I = \frac{m(g-a)r}{\alpha} = \frac{m\left(g - \frac{2h}{t^2}\right)r}{2h/rt^2}$$

$$= m\left(g - \frac{2h}{t^2}\right)\frac{r^2t^2}{2h}$$

Ex.12 The M.I. of rod of length 1.5 m and mass 0.1 kg about the axis passing through centre and perpendicular to the length of rod is I_1 and that through one end perpendicular to rod is I_2. Now $I_1 \times I_2$ will be (in $kg^2\ m^4$)

(A) .15 (B) 0.25

(C) .75 (D) .35

Sol.(A) $I_1 = \frac{M\ell^2}{12} = \frac{0.1 \times (1.5)^2}{12} = 0.01875\ kg.m^2$

$I_2 = \frac{M\ell^2}{3} = \frac{0.1 \times (1.5)^2}{3} = 0.075\ kg.m^2$

Now, $I_1 \times I_2 = 1.4 \times 10^{-3}\ kg^2.m^4$

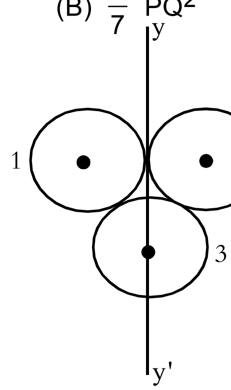
Ex.13 Three rings, each of mass P and radius Q are arranged as shown in fig.

The moment of inertia of the arrangement about yy' axis will be-

(A) $\frac{7}{2}$ PQ^2 (B) $\frac{2}{7}$ PQ^2

(C) $\frac{2}{5}$ PQ^2

(D) $\frac{5}{2}$ PQ^2



Sol.(A)

M.I. of ring '1' about yy' = M.I. of ring about the tangent parallel to its plane

$\Rightarrow I_1 = (3/2)\ MR^2$

Similarly, M.I. of ring '2' about yy', $I_2 = (3/2)\ MR^2$

M.I. of ring '3' about yy' = M.I. of ring about its diameter

$\Rightarrow \quad I_3 = \frac{MR^2}{2}$

Now M.I. about yy' is $I = I_1 + I_2 + I_3 = (7/2)\ MR^2 = (7/2)\ PQ^2$

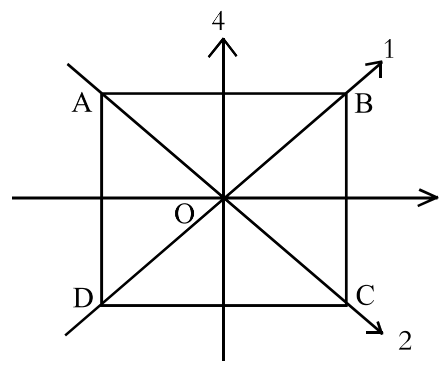
Ex.14 Select the correct alternative(s). The moment of inertia I of a thin square plate ABCD (figure) of uniform thickness about an axis passing through the centre O and perpendicular to the plane of the plate is -

(A) $I_1 + I_2$ (B) $I_3 + I_4$

(C) $I_1 + I_3$ (D) $I_1 + I_2 + I_3 + I_4$

(where I_1, I_2, I_3 and I_4 are respectively the moments of inertia about axes 1, 2, 3 and 4 which are in the plane of the plate.)

Sol.(A, B, C)



By the theorem of perpendicular axes,

$I = I_1 + I_2 = I_3 + I_4$

By symmetry : $I_1 = I_2$ and $I_3 = I_4$

$\therefore \quad I = 2I_1 = 2I_3$

or $\quad I_1 = I_3$.

Thus $\quad I_1 = I_2 = I_3 = I_4$.

$\therefore \quad I = I_1 + I_2 = I_1 + I_3 = I_3 + I_4$

Ex.15 Given that, $\vec{r} = 2\hat{i} + 3\hat{j}$ and $\vec{F} = 2\hat{i} + 6\hat{k}$. The magnitude of torque will be-

(A) $\sqrt{405}$ N.m (B) $\sqrt{410}$ N.m

(C) $\sqrt{504}$ N.m (D) $\sqrt{510}$ N.m

Sol.(C) We know that, $\vec{\tau} = \vec{r} \times \vec{F}$

$\Rightarrow \vec{\tau} = (2\hat{i} + 3\hat{j}) \times (2\hat{i} + 6\hat{k})$

$= 12(-\hat{j}) + 6(-\hat{k}) + 18\hat{i}$

$= -12\hat{j} - 6\hat{k} + 18\hat{i}$

[Note: $\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = k$, $\hat{j} \times \hat{i} = -k$ etc]

Now $|\vec{\tau}| = \sqrt{(-12)^2 + (-6)^2 + (18)^2}$

$= \sqrt{144+36+324} = \sqrt{504}$

Ex.16 A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 sec. The value of torque will be-

(A) $4A_0$ (B) $12A_0$

(C) A_0 (D) $\frac{3A_0}{4}$

Sol.(D) Torque, τ = rate of change of angular momentum

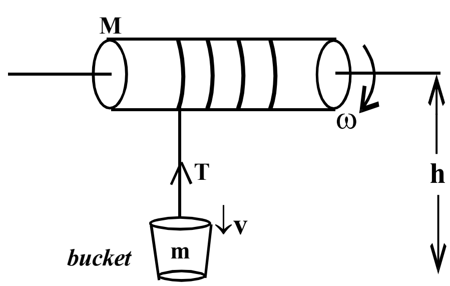
$= \frac{dJ}{dt} = \frac{4A_0 - A_0}{4} = \frac{3A_0}{4}$

Ex.17 A cylinder of mass M and radius r is mounted on a frictionless axle over a well. A rope of negligible mass is wrapped around the cylinder and a bucket of mass m is suspended form the rope. The linear acceleration of the bucket will be-

(A) $\frac{mg}{M+m}$ (B) $\frac{mg}{M+2m}$

(C) $\frac{2mg}{M+2m}$ (D) $\frac{M}{mg}$

Sol.(C)



$mg - T = ma$

$T = m(g - a)$ ....(1)

$\tau = I\alpha = rT \Rightarrow (1/2)\, Mr^2 .\, a/r = rT$

$\Rightarrow T = 1/2\, Ma$ ......(2)

From (1) and (2),

$(1/2)\, Ma = m(g - a) \Rightarrow \frac{M+2m}{2} a = mg,$

$\Rightarrow a = \frac{2mg}{M+2m}$

Ex.18 **In the example 17**, the velocity of bucket, after it has fallen through a height h will be-

(A) $\left(\frac{4mgh}{M+2m}\right)^{1/2}$ (B) $\left(\frac{4Mgh}{M+2m}\right)^{1/2}$

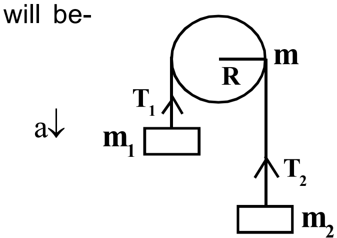
(C) $\left(\frac{2mgh}{M+2m}\right)^{1/2}$ (D) $\left(\frac{mgh}{M+m}\right)^{1/2}$

Sol.(A) Let v be the velocity of bucket after falling a height h.

Now, using $v^2 = u^2 + 2ah$

$v = \sqrt{2ah} = \sqrt{\frac{2.2mgh}{(M+2m)}} = \left(\frac{4mgh}{M+2m}\right)^{1/2}$

Ex.19 Two bodies of masses m_1 and m_2 are attached to the two ends of a string. The string passes over a pulley of mass m and radius R. If $m_1 > m_2$, the acceleration of the system will be-



(A) $\frac{m_1 - m_2}{m_1 + m_2} g$ (B) $\frac{m_1 + m_2}{m_1 - m_2} g$

(C) $\frac{m_1 - m_2}{m_1 + m_2 + (1/2)m}$ (D) $\frac{m_1 - m_2 + m}{m_1 + m_2 + m}$

Sol.(C) $m_1 g - T_1 = m_1 a$ ....(i)

$T_2 - m_2 g = m_2 a$ ....(ii)

$\therefore (T_1 - T_2) + (m_1 + m_2)\, a = (m_1 - m_2)\, g$

$\Rightarrow T_1 - T_2 = (m_1 - m_2)\, g - (m_1 + m_2)\, a$

$\tau = (T_1 - T_2)\, R = I\alpha$

$= (1/2)\, mR^2\, (a/R)$ ....(iii)

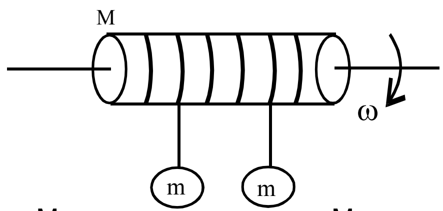
$\Rightarrow (T_1 - T_2) = (1/2)\, ma$ ....(iv)

From (iii) and (iv)

$(1/2)\, ma = (m_1 - m_2)\, g - (m_1 + m_2)\, a$

$$\Rightarrow \qquad a = \frac{m_1 - m_2}{m_1 + m_2 + (m/2)} g$$

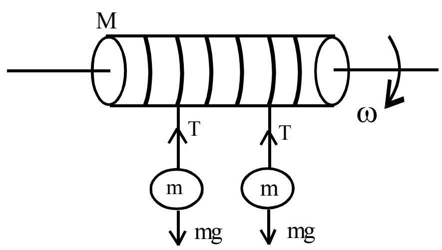
Ex.20 A uniform solid cylinder of mass m and radius R rotates about a frictionless horizontal axle. Two similar masses suspended with the help of two ropes wrapped around the cylinder. If the system is released from rest then the tension in each rope will be-



(A) $\frac{Mmg}{(M+m)}$ (B) $\frac{Mmg}{(M+2m)}$

(C) $\frac{Mmg}{(M+3m)}$ (D) $\frac{Mmg}{(M+4m)}$

Sol.(D)



mg – T = ma

mg – T = ma

From these equation,

2mg – 2T = 2ma ......(i)

$\tau = (2T)\,R = I\,\alpha = (1/2)MR^2.\ (a/R)$ .....(ii)

From (i) and (ii), $T = \frac{Mm}{M+4m} g$

Note : Also a = 4mg (M + 4m)

Ex.21 A body whose moment of inertia is 3 kg. m^2, is at rest. It is rotated for 20 seconds with moment of force 6N-m. The work done will be- (in joule)

(A) 24 (B) 240

(C) 2400 (D) 24000

Sol.(C) Suppose an angular acceleration α is produced in the body by applying a torque τ. Then,

$\tau = I\alpha$, where I is the moment of inertia of body about the axis of rotation.

Hence τ = 6 N.m and I = 3 kg.m^2

Therefore, $\alpha = \frac{\tau}{I} = \frac{6}{3}$ = 2 rad/sec^2

Initially, the body is at rest ($\omega_0 = 0$). It rotates under the action of angular acceleration α for 2 seconds. In this time-interval, the angular displacement of the body is,

$\theta = \omega_0 t + (1/2)\,\alpha\, t^2$

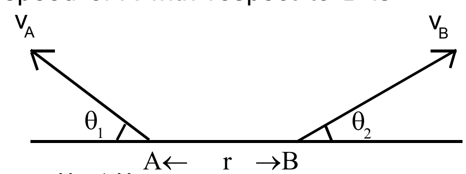
$= 0 + (1/2) \times 2 \times (20)^2 = 400$ radian

Work done in this displacement,

W = (torque × displacement) = $\tau \times \theta$

= 6 × 400 = 2400 joule

Ex.22 Two particles A and B are moving as shown in fig. At this moment of time the angular speed of A with respect to B is-



(A) $\frac{v_A + v_B}{r}$

(B) $\frac{v_A - v_B}{r}$

(C) $\frac{v_B \sin\theta_2 - v_A \sin\theta_1}{r}$, in anticlockwise direction

(D) $\frac{v_B \sin\theta_2 + v_A \sin\theta_1}{r}$, in anticlokwise direction

Sol.(C) Resolving the velocities along x and y axes, we have,

$v_x = -\,v_A \cos\theta_1$

$v_y = v_A \sin\theta_1$and $v'_x = v_B \cos\theta_1$,

$v'_y = v_B \sin\theta_2$

The relative velocity of A with respect to B along y-axis is given by ($v_A \sin\theta_1 - v_B \sin\theta_2$)

$\because\ \vec{v} = \vec{r} \times \vec{\omega}$, Because v is in positive y direction and position vector of A w.r.t. B is in negative x direction so the direction of ω will be in clock wise and magnitude of ω will be

$\omega = \left(\frac{v_A \sin\theta_1 - v_B \sin\theta_2}{r}\right)$ clockwise or

$\omega = \left(\frac{v_B \sin\theta_2 - v_A \sin\theta_1}{r}\right)$ anticlockwise

Ex.23 A particle is moving in x-y plane and the components of its velocity along x and y axis are V_x and V_y. The angular momentum about the origin will be-

(A) $m\hat{k}\,(xV_y - yV_x)$ (B) $\frac{\hat{k}}{2}(xV_y - yV_x)$

(C) $m\hat{k}\sqrt{xV_y - yV_x}$ (D) $\frac{\hat{k}}{2}\sqrt{xV_y - yV_x}$

Sol.(A) We know that angular momentum of a particle

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

$$= m\begin{vmatrix} i & j & k \\ x & y & z \\ V_x & V_y & V_z \end{vmatrix} = m\begin{vmatrix} i & j & k \\ x & y & 0 \\ V_x & V_y & 0 \end{vmatrix}$$

$$= m\hat{k}\,(xV_y - yV_x)$$

Ex.24 A ring of mass 10 kg and diameter 0.4 m is rotating about its geometrical axis at 1200 rotation perminute. Its moment of inertia and angular momentum will respectively be-

(A) 0.4 kg-m^2, 50.28 J-sec
(B) 50.24 kg-m^2, 0.4 J-sec
(C) 0.4 J-sec, 50.24 kg-m^2
(D) 0.4 kg-m^2, 0

Sol.(C) M.I of a ring about its geometrical axis = M.I. of ring about an axis passing through its centre and perpendicular to its plane.

$= MR^2 = 10\,(0.2)^2 = 10 \times 0.04 = 0.4$ kg-m^2

Now Angular momentum, $J = I.\omega = I.\frac{2\pi n}{t}$

$= 0.4 \times \frac{2\pi \times 1200}{60} = 50.24$ J-sec

Ex.25 A cockroach of mass m is moving on rim of a disc of radius r with velocity v in anticlockwise direction. The moment of inertia of the disc about its own axis is I and it is rotating in the clockwise direction with angular speed ω. If the cockroach stops moving then the angular speed of the disc will be-

(A) $\frac{I\omega}{I + mR^2}$ (B) $\frac{I\omega + mvr}{I + mr^2}$

(C) $\frac{I\omega - mvr}{I + mr^2}$ (D) $\frac{I\omega - mvr}{I}$

Sol.(C) According to law of conservation of angular momentum, Angular momentum before the cockroach stops

= Angular momentum after the cockroach stops.

$\Rightarrow \quad I\omega - mvr = (I + mr^2)\,\omega'$

$\therefore \quad \omega' = \frac{I\omega - mvr}{I + mr^2}$

Ex.26 A solid cylinder of mass 2 kg and radius 0.2 m is rotating about its own axis without friction with angular velocity 3 rad/sec. A particle of mass 0.5 kg and moving with a velocity of 5 m/s strikes the cylinder and sticks to it as. The loss in energy due to collision will be-

(A) 6.25 J (B) 5.25 J
(C) 4.25 J (D) 3.25 J

Sol.(D) According to conservation of angular momentum,

Angular momentum before collision = Angular momentum after collision ....(i)

Angular momentum of cylinder befor callision

$J_1 = I\omega = (1/2)\,mR^2\,\omega$

$= (1/2) \times 2 \times 0.04 \times 3 = 0.12$ J.sec

Now from (i)

$J_{cy\ell} + m_p vR = (I + mR^2)\,\omega$

$\Rightarrow \omega = \frac{0.12 + 0.5 \times 5 \times 0.2}{(1/2) \times 2 \times 0.04 + 0.5 \times 0.04} = 10.3$ rad/sec

Now energy of system before collision

$E = (1/2)\,I\omega^2 + (1/2)\,mv^2$

$= (1/2) \times (1/2) \times 2 \times 0.04 \times 9 + (1/2) \times 0.5 \times 25 = 6.43$ J

Energy of system after collision

$E' = (1/2)\,I'\omega'^2 = (1/2) \times (1/2\,M + m)\,R^2\omega^2$

$= (1/2) \times (1/2 \times 2 + 0.5) \times 0.04 \times (10.32)^2$

$= 3.18$ J

Now $E - E_1 = 6.43 - 3.18 = 3.25$ J

Ex.27 A particle of mass 3 kg is moving under the action of a central froce whose potential energy is given by $U(r) = 10\,r^3$ joule. For what energy and angular momentum will the orbit be a circle of radius 10m ?

(A) 2.5×10^4J, 3000 kg m^2/sec
(B) 2.5×10^3J, 3000 kg m^2/sec
(C) 2.5×10^2J, 30000 kg m^2/sec
(D) 2.5×10^2J, 300 kg m^2/sec

Sol.(A) The (variable) potential energy of the particle, $U(r) = 10r^3$ joule. Hence the force acting on the particle is

$$F = -\frac{\partial U}{\partial r} = -30r^2$$

For circular motion of the particle,

$$F = \frac{mv^2}{r} = 30r^2$$

Substituting m = 3 kg and r = 10 m,
we get v = 100 m/s

Total energy of the particle in circular motion is

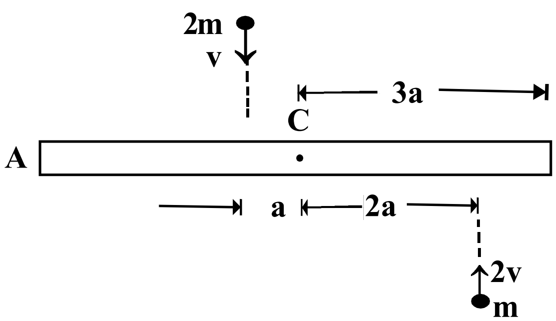
$$E = \frac{1}{2} mv^2 + U(r)$$

$$= \frac{1}{2} \times 3 \times (100)^2 + 10 \times (10^3) = 2.5 \times 10^4 \text{ J}$$

Angular momentumof the particle is

$J = mvr = 3 \times 100 \times 10 = 3000$ kg m^2/s

Ex.28 Select the correct choice(s) : mass 8 m lies on a smooth horizontal table. Two point masses m and 2 m moving in the same horizontal plane with speeds 2v and v respectively strike the bar as shown in figure and stick the bar after collision. Denoting angular velocity (about the centre of mass,) total energy and centre of mass velocity by ω, E and v_c respectively, we have after collision ‑



(A) $v_c = 0$ (B) $\omega = 3v/5a$,
(C) $\omega = v/5a$ (D) $E = 3mv^2/5$

Sol.(A, C, D)

Both the masses, 2m and m, after striking the bar give equal momentum each 2mv to the bar in oppoisite directions. Hence, after collision the bar has no translational motion, i.e. the linear velocity of the centre of mass of the bar is zero ($v_c = 0$).

When both the masses stick to the bar, the whole system rotates about the centre of mass C. As there is no external torque acting on the system, the angular momentum is conserved.

Before collision (bar is stationary), there will be only the angular momenta of 2m and m about C. Hence the initial angular momentum (see Fig.) is

$J_i = 2mva + m(2v)2a = 6mva$

(in an anticlock wise direction)

After collision, the bar and both the masses (2m and m) rotate with angular velocity ω about the centre of mass C. The moment of inertia of the bar (mass 8m and lenght 6a) about C is

$\frac{M\ell^2}{12} = \frac{8m(6a)^2}{12} = 24\ ma^2$, and the moments of inertia of 2 m and m about C are 2 ma^2 and 4 ma^2 respectively. Hece, after collision the angular momentum of the system is

$J_f = (24\ ma^2 + 2\ ma^2 + 4\ ma^2)\ \omega = 30\ ma^2\omega$

But $J_i = J_f$

(conservation of angular momentum)

$$\omega = \frac{6mav}{30ma^2} = \frac{v}{5a}$$

(in an anticlock wise direction)

The angular kinetic energy

$\left(\frac{1}{2}I\omega^2\right)$ of the system is

$$= \frac{1}{2}(30\ ma^2)\left(\frac{v}{5a}\right)^2 = \frac{3}{5}mv^2$$

Ex.29 A cord is wound round the circumference of a wheel of radius r. The axis of the wheel is horizontal and moment of inertia about it is I. A weight mg is attached to end of the cord and falls from rest. After falling through a distance h, the angular velocity of the wheel will be-

(A) $\sqrt{\frac{2gh}{I+mr}}$ (B) $\sqrt{\frac{2mgh}{I+mr^2}}$

(C) $\sqrt{\frac{2mgh}{I+2m}}$ (D) $\sqrt{2gh}$

Sol.(B) $mgh = (1/2)\ I\omega^2 + (1/2)\ mv^2$

$= (1/2)\ I\omega^2 + (1/2)\ mr^2\omega^2$

or $2mgh = [I + mr^2]\omega^2$,

$$\therefore \quad \omega = \left[\frac{2mgh}{I + mr^2}\right]^{1/2}$$

Ex.30 A mass m is supported by a massless string wound round a uniform cylinder of mass m and radius R. On releasing the mass from rest, it will fall with acceleration-

(A) g (B) g/2
(C) g/3 (D) 2g/3

Sol.(D) $mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$

$$= \frac{1}{2} mv^2 + \frac{1}{2}\left[\frac{1}{2} mR^2\right] v^2/R^2 = \frac{3}{4} mv^2$$

$$v = \sqrt{2ah} \quad [\because v^2 = u^2 + 2as]$$

$$\therefore \quad mgh = \frac{3}{4} m \times 2ah \quad \Rightarrow a = \frac{2}{3} g$$

Ex.31 Let g be the acceleration due to gravity at earth's surface and K be the rotational kinetic energy of the earth. Suppose the earth's radius decreases by 2%, keeping all other quantities same, then-

(A) g decreases by 2% and K decreases by 4%
(B) g decreases by 4% and K increases by 2%
(C) g increases by 4% and K decreases by 4%
(D) g decreases by 4% and K increases by 4%

Sol.(C) We know that that $g = \frac{GM}{R^2}$

take logarithmic on both side :

$\log g = \log GM - 2\log R$

$\Rightarrow \log g = \log G + \log M - 2 \log R$

Now partially differentiating on both side

$$\frac{\Delta g}{g} = 0 + \frac{\Delta M}{M} - 2\frac{\Delta R}{R}$$

so the percentage change in g

$$\therefore \frac{\Delta g}{g} \times 100 = \frac{\Delta M}{M} \times 100 - 2\frac{\Delta R}{R} \times 100$$

Because radius decreases by 2% and other quantities are constant ($\therefore \Delta M = 0$) so

$$\frac{\Delta g}{g} \times 100 = 0 - \frac{2\left(\frac{-2}{100}R\right)\times 100}{R} = 4\%$$

Simillarly for rotational kinetic energy

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{2}{5} MR^2\right)\omega^2 = \frac{1}{5} MR^2\omega^2$$

So for the percentage change

$$\Rightarrow \frac{\Delta K}{K} \times 100 = \frac{\Delta M}{M} \times 100 + 2\frac{\Delta R}{R} \times 100 + 2\,\frac{1}{5}\frac{\Delta W}{W} \times 100$$

$$\Rightarrow \Delta M = 0,\ \Delta R = \left(\frac{-2}{100}R\right) \text{ and } \Delta W = 0$$

$$\therefore \frac{\Delta K}{K} \times 100 = -4\%$$

(so K.E. is decreasing by 4%)

Ex.32 When a solid sphere of moment of inertia I rolls down an inclined plane, then the percentage of its rotational kinetic energy is-

(A) 100% (B) 50%
(C) 28% (D) 72%

Sol.(C) M.I. of sphere about the diameter, $I = \frac{2}{5} mr^2$

The rotational kinetic energy

$$K_r = (1/2)\ I\omega^2 = (1/5)\ mr^2\omega^2$$

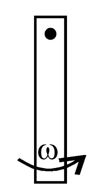
The translational kinetic energy

$$K_t = (1/2)\ mv^2 = (1/2)\ m.r^2\omega^2$$

Total energy , $K = K_r + K_t = \frac{7}{10} mr^2\omega^2$

Now, $\frac{K_r}{K} \times 100\% = \frac{1/5}{7/10} \times 100\% = 28\%$

Ex.33 A uniform thin rod of length ℓ is suspended from one of its ends and is rotated at f rotations per second. The rotational kinetic energy of the rod will be-



(A) $(2/3)\ \pi^2 f^2\ m\ell^2$
(B) $(4/3)\ f^2\ m\ell^2$
(C) $4\pi^2 f^2 m\ell^2$
(D) 0

Sol.(A) The M.I. of a rod about an axis passing through its one ends

and perpendicular its to axis I, $= \frac{M\ell^2}{3}$

Now rotational kinetic energy $K_r = (1/2)\ I\omega^2$

$$= (1/2)\ \frac{M\ell^2}{3} . (2\pi f)^2$$

$$= (2/3)\ M\ell^2.\pi^2 f^2$$

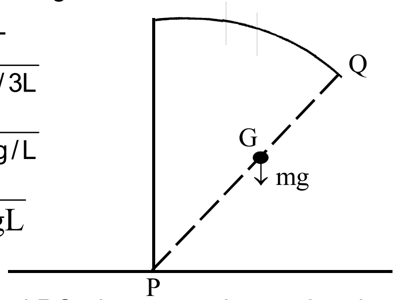
Ex.34 A thin uniform rod PQ of mass M and length L is free to rotate about a hinge at P in the floor. Initially the rod is vertical. If it is released from this position then its angular velocity while striking the floor will be-

(A) 3g/L

(B) $\sqrt{g/3L}$

(C) $\sqrt{3g/L}$

(D) $\sqrt{gL}$



Sol.(C)

M.I.of rod PQ about an axis passing through the point P and perpendicular to the rods is

$$I = \frac{mL^2}{3}$$

Now according to conservation of energy,

$$Mgh = (1/2)\ I\omega^2$$

$$\Rightarrow \quad m.g.\ (1/2)\ L = (1/2)\ \frac{mL^2}{3}\ \omega$$

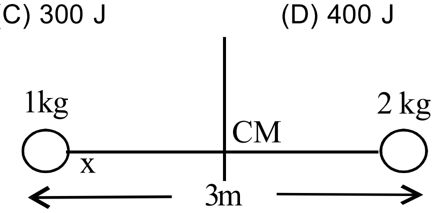
$$\Rightarrow \quad \omega = \sqrt{\frac{3g}{L}}$$

Ex.35 Two bodies of mass 1 kg and 2 kg are attached to the two ends of a 3 m long rod. This rod is rotating about an axis passing through centre of mass with angular velocity 10 rad/sec and perpendicular to its length. The rotational K.E. of the system will be-

(A) 150 J (B) 755 J

(C) 300 J (D) 400 J

Sol.(D)



Suppose the C.M. of the system be at x distance from 1 kg.

Now $\sum$ moment of masses about C.M. = 0

$\Rightarrow$ 1.x – 2.(3 – x) = 0 or x = 2

Now C.M. will be at distance 2 m from 1 kg and 1 m from 2 kg

Now M.I. of 1 kg mass about an axis passing through

C.M. and $\perp$ rod, $I_1 = 1.x^2 = 1.2^2 = 5$ kg-m^2

Similarly, for 2kg mass, $I_2 = 2.1^2 = 3$ kg m^2

Net M.I., $I = I_1 + I_2 = 8$ kg m^2

Now rotational

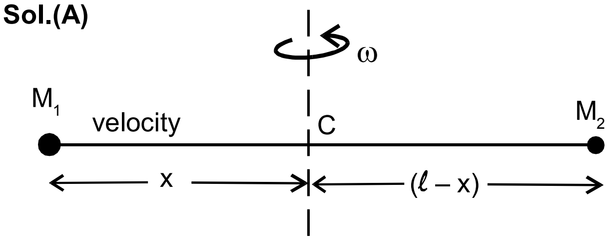
K.E. = (1/2) $I\omega^2$ = (1/2) × 8 × 100 = 400 J

Ex.36 Point masses M_1 and M_2 are placed at the ℓ and negligible mass. The rod is to be set rotating about an axis perpendicular to its lenght . Locate a point on the rod through which the axis of rotation should pass in order that the work required to set the rod rotating with angular ends of a rigid rod of lenght ω is minimum.

(A) $\frac{M_2}{M_1+M_2}\ell$ (B) $\frac{M_1}{M_1+M_2}\ell$

(C) $\frac{M_1}{M_1-M_2}\ell$ (D) $\frac{M_2}{M_1-M_2}\ell$

Sol.(A)



Suppose the rod is set rotating about an axis passing through a point C, distatnt x from M_1. The moment of inertia of the system about this axis is given by

$$I = M_1x^2 + M_2(\ell - x)^2$$

The work done in setting the rod rotating with angular velocity ω equals the kinetic energy of rotation acquired by the rod. That is,

$$W = \frac{1}{2}\ I\omega^2 = \frac{1}{2}\ [M_1x^2 + M_2(\ell - x)^2]\ \omega^2$$

For W to be minimum, we must have $\frac{dW}{dx} = 0$

or $2M_1x - 2M_2(\ell - x) = 0$

or $$x = \frac{M_2}{M_1+M_2}\ell$$